International Journal of Applied Mathematics &
Statistical Sciences (IJAMSS)
ISSN 2319-3972
Vol. 2, Issue 1, Feb 2013, 49-62

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

PERIOD DOUBLING ROUTE IN THE PERIODIC AND THE CHAOTIC REGION OF THE LOGISTIC MAP

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ABSTRACT

One of the common route to chaos is the period doubling route [6, 17]. For systems that undergo period doubling cascades, there also exists an “inverse cascade” [2] of chaotic band merging called reverse bifurcation [4, 8, 10]. This paper investigates the period doubling route to chaos and the period doubling nature of chaotic bands using the logistic map. We have considered this map and identified the parameter values μ for which the period doubling bifurcations occur and have shown that the bifurcation points converges to an accumulation point where the chaotic situation starts. Our tool for finding such a point is with the help of establishing the ‘Feigenbaum delta’ [3] which is one of the several universalities discovered by famous particle physicist M. J. Feigenbaum. The period doubling scenario explains us how the behaviour of the model changes from regularity to a chaotic one. Further, we have discussed about the reverse bifurcation and reverse bifurcation points called Misiurewicz points [13,14, 15, 18] and established the Feigenbaum delta in that case also. This situation occurs inside the chaotic region and it unfolds some regularity even within the chaotic region.

KEYWORDS: Period Doubling Bifurcation, Reverse Bifurcation, Feigenbaum Delta, Chaos

INTRODUCTION

One generally thinks that a chaotic systems needs a complicated formula for its mathematical representation. But, there are very simple functions which can lead not only to chaos, but can make us understand how this chaotic situation gets developed from ‘ordered’ behaviour. The logistic function, used in population dynamics, is one of these functions. The logistic function is

$$f(x) = \mu x(1 - x) \tag{1}$$

where μ indicates the “fertility” or “growth rate” of a population with limited resources. Here, $0 \leq \mu \leq 4$ and $x \in [0,1]$, the reason of which are well discussed in [6, 12]. We can see that this function represents an inverted parabola (Figure 1), intersecting the x axis in $(0,0)$ and $(1,0)$.

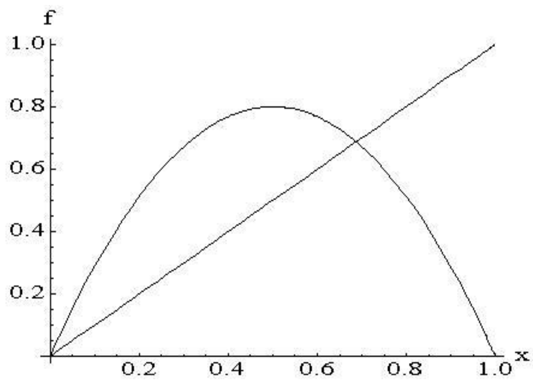

Figure 1

For values $0 \leq \mu \leq 4$, the height of the parabola will be in the interval [0, 1]. If we iterate this function, we will observe the discrete dynamics of the population that the function models.

Over the last more than twenty five years, the logistic map has served as an example to understand nonlinear dynamics and chaos. As R. M. May stressed some 36 years ago, the patterns formed by iterates of the logistic map are simple to compute but illustrate the complexities possible in nonlinear dynamics [12]. The bifurcation of the logistic map, which summaries the long- time dynamics as a function of the control parameter μ in equation 1, is one of the most commonly reproduced images of dynamical systems.

Also, using this map, M.J. Feigenbaum derived his famous renormalization-group theory of scaling exponents [3].

The universality discovered by Feigenbaum [3] with nonlinear models has successfully led to observe that large classes of nonlinear systems exhibit transition to chaos through period doubling route. In 1999, Kuruvila and Nandakunmaran [9] restrained chaos in semiconductor laser using reverse bifurcation. So there is not only theory value but also application value in the study of Period Doubling and Reverse Bifurcation.

PERIOD DOUBLING BIFURCATION AND PERIOD DOUBLING ROUTE TO CHAOS

In a period-doubling bifurcation, the previously stable fixed/periodic points become unstable after attaining some value of the control parameter, and stable periodic trajectories of period, doubled to the previous one appears near it. The original fixed/periodic points continues to exist as unstable fixed/periodic points, and all the remaining points are attracted towards the new stable periodic trajectories of doubled period. This process repeats itself up to certain critical value of the parameter where chaos creeps into the previously deterministic system.

The logistic map $f(x) = \mu x(1-x)$ has two fixed points 0 and $1-\frac{1}{\mu}$ which are solutions of the equation $f(x) = x$.

Since $\frac{df}{dx} = \mu(1-2x)$, hence the derivative of $f(x)$ at these fixed points are:

$$\left.\frac{df}{dx}\right|_{x=0} = \mu < 1 \text{ for } \mu < 1.$$

That is, the fixed point 0 is an attracting or stable fixed point for $\mu < 1$. Also, as

$$\left.\frac{df}{dx}\right|_{x=1-\frac{1}{\mu}} = 2-\mu > 1 \text{for } \mu < 1.$$

the fixed point $1-\frac{1}{\mu}$ is a repelling or unstable fixed point for $\mu < 1$

So, it is seen that at $\mu = 1$, a transcritical bifurcation takes place because for $\mu > 1$ the two fixed points exchange their stability i.e., the fixed point 0 becomes an unstable fixed point and the fixed point $1-\frac{1}{\mu}$ becomes a stable fixed point. At $\mu = 3$, the fixed point $1-\frac{1}{\mu}$ loses its stability. Hence the region of stability for the fixed point $1-\frac{1}{\mu}$ is $1 < \mu < 3$.

If we increase μ beyond 3, the fixed point $1-\frac{1}{\mu}$ becomes unstable.

At the parameter value $\mu = 3$, $\left.\frac{df}{dx}\right|_{x=1-\frac{1}{\mu}} = -1$. This shows that after the parameter value $\mu = 3$ both the fixed points found from the equation $f(x) = \mu x(1-x) = x$ ceases to be stable.

We now consider the iterated map $f^2(x)$.

The fixed point of the second iterated map is given by

$$f^2(x) = x \Rightarrow -\mu^3x^4 + 2\mu^3x^3 - (\mu^2+\mu^3)x^2 + (\mu^2-1)x = 0$$

Solving the above equation we get four solutions, viz.

$$0, \quad 1-\frac{1}{\mu}, \quad \frac{\mu+1+\sqrt{\mu^2-2\mu-3}}{2\mu}, \quad \frac{\mu+1-\sqrt{\mu^2-2\mu-3}}{2\mu}$$

Out of the above four solutions the first two are fixed points of f, which already became unstable after the control parameter μ attains the value 3. The other two solutions, the fixed points of $f^2(x)$, are the periodic point of period two of the logistic map f.

As we are considering only the real solutions, for the two fixed points

$$x_1 = \frac{\mu+1+\sqrt{\mu^2-2\mu-3}}{2\mu}, \qquad x_2 = \frac{\mu+1-\sqrt{\mu^2-2\mu-3}}{2\mu}$$

we must have $\mu \geq 3$. Hence, for μ values between 0 and 4, these solutions are defined only for $\mu \geq 3$. Moreover at $\mu = 3$, we set $x_1 = x_2 = \frac{\mu-1}{\mu}$ i.e., the two solutions bifurcate from the fixed point $1-\frac{1}{\mu}$. Figure 2 shows a bifurcation diagram at $\mu = 3$.

Thus, we say that at $\mu = 3$, The logistic map trajectories undergo period doubling bifurcations. Just below $\mu = 3$, the trajectories converge to a single value of x. Just above $\mu = 3$, the trajectories tend to alter between two values of x.

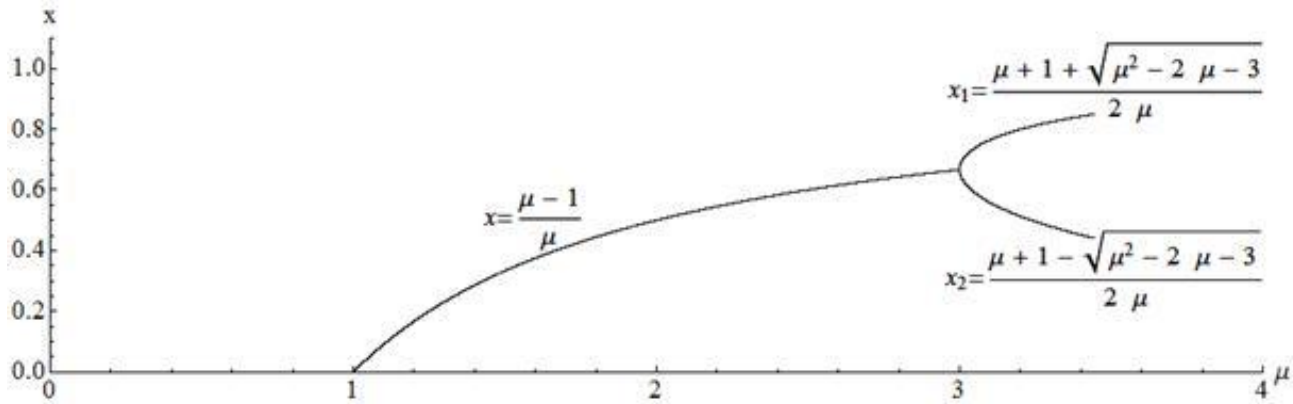

Figure 2: The Period Doubling Bifurcation at $\mu = 3$

Let us see how the derivatives of the map function and of the second iterate function change at the bifurcation value. Equation $\left.\frac{df}{dx}\right|_{x=\frac{\mu-1}{\mu}} = 2-\mu$ tells us that $\frac{df}{dx}$ passes through the value -1 as μ increases through 3. Next we can evaluate the derivative of the second iterate function by using the chain-rule of differentiation.

$$\frac{df^{(2)}(x)}{dx} = \frac{df\big(f(x)\big)}{dx} = \left.\frac{df}{dx}\right|_{f(x)} \left.\frac{df}{dx}\right|_{x}$$

If we now evaluate the derivative at one of the fixed points say, x_1, we find

$$\left.\frac{df^{(2)}(x)}{dx}\right|_{x_1} = \left.\frac{df}{dx}\right|_{x_2} \left.\frac{df}{dx}\right|_{x_1} = \left.\frac{df^{(2)}(x)}{dx}\right|_{x_2} \tag{2}$$

In arriving at the last result we made use of the fact that $x_2 = f(x_1)$ for the two periodic points of period 2. Equation (2) states a rather surprising and important result-

The derivative of $f^{(2)}$ are the same at both the fixed points that are part of the two cycles. This result tells us that both these fixed points are stable or both are unstable and that they have the same 'degree' of stability or instability. Again, since the derivative of $f(x)$ is equal to -1 for $\mu = 3$, equation (2) tells us that the derivative of $f^{(2)}$ is equal to $+1$ for $\mu = 3$. As μ increases further, the derivative of $f^{(2)}$ decreases and the fixed points become stable.

For μ just grater than 3, we see that the slope of $f^{(2)}$ at those two fixed points is less than 1 and hence they are stable fixed points of $f^{(2)}$. Besides this, the unstable fixed point of $f(x)$ located at $1-\frac{1}{\mu}$ is also an unstable fixed point of $f^{(2)}(x)$. The two 2-cycle fixed points of $f^{(2)}$ continue to be stable fixed points until $\mu = 1+\sqrt{6}$. At this value of μ, which is denoted by μ_2, the derivative of $f^{(2)}$ evaluated at the two cycle fixed points is equal to -1 and for values of μ larger than μ_2, the derivative is more negative than -1. Hence for μ values greater than μ_2, the 2-cycle points are unstable fixed points. We find that for μ values just greater than μ_2, the trajectories settle into a 4 cycle, i.e. the trajectory cycles among 4 values which we can label as x_1^*, x_2^*, x_3^* and x_4^*. To determine these periodic points we need to solve an eight degree equation viz. $f^4(x) = x$ which is manually cumbersome and time consuming. With the help of computer programs mentioned in detail in [20], we obtained the bifurcation points and the value of Feigenbaum delta which are furnished below:

Table 1

Bifurcation Points	Periods
$\mu_1 = 3$	Period 2 is born
3.44948974278317…	Period 4 is born
3.54409035955192 ...	Period 8 is born
3.56440726609543 ...	Period 16 is born
3.56875941954382 ...	Period 32 is born
3.56969160980139 ...	Period 64 is born
3.56989125937812 ...	Period 128 is born
3.56993401837397 ...	Period 256 is born
3.5699431760484 ...	Period 512 is born
3.56994513734217 ...	Period 1024 is born

Based on these bifurcation values, we compute

$$\delta_1 = \frac{\mu_2 - \mu_1}{\mu_3 - \mu_2} = 4.751446218177\ldots,$$

$$\delta_2 = \frac{\mu_3 - \mu_2}{\mu_4 - \mu_3} = 4.656251017651\ldots,$$

$$\delta_3 = \frac{\mu_4 - \mu_1}{\mu_5 - \mu_4} = 4.668242235582\ldots,$$

$$\delta_4 = \frac{\mu_5 - \mu_4}{\mu_6 - \mu_5} = 4.668739469275\ldots$$

$$\delta_5 = \frac{\mu_6 - \mu_7}{\mu_8 - \mu_9} = 4.669132150630\ldots$$

and so on.

Then the Feigenbaum delta is evaluated as

$$\delta = \lim_{k \to \infty} \delta_k = 4.669201609\ldots$$

The nature of δ is universal i.e. it is the same for a wide range of different iterations.

BIFURCATION DIAGRAM

The bifurcation diagram [6, 17] has been one of the frequently used tools to study the chaotic or periodic behaviour of one dimensional maps. We can summarize the behaviour of the logistic map with the help of a bifurcation diagram.

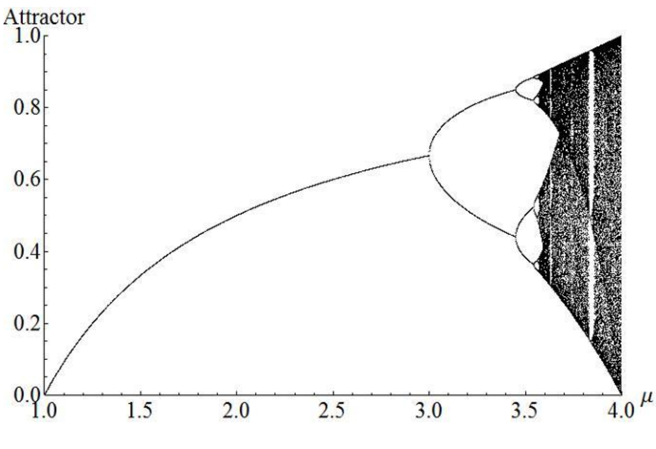

Figure 3

From our previous discussion it was seen that for $1 < \mu < 3$, $1 - \frac{1}{\mu}$ is the stable point attractor and its value increases as the value of μ increases. For $\mu > 3$, the attractor is a period-2 cycle, as indicated by the two branches. As μ increases, both branches split simultaneously, yielding a period 4 cycle. This splitting is the period doubling bifurcation. A cascade of further period-doublings occurs as μ increases, yielding period-8, period-16, and so on, until at $\mu = \mu_\infty \approx 3.56994567\ldots$, the map becomes chaotic and the attractor changes from a finite to an infinite set of points [19].

A FEW OBSERVATIONS HAS BEEN MADE ON THE BIFURCATION DIAGRAM

First Observation

In the formation of bifurcation diagram we observed that in the periodic region of the bifurcation diagram when we plot the iterations (with respect to the parameter after attaining the super cycle values) the odd iterations form the upper branch whereas the even iterations form the lower branch of the bifurcation diagrams [1, 7, 11].

If we symbolise the successive iterations by the numbers 1,2,3,4,5,6,… ,considering the first odd iterations as 1 we found that the iterate labels from top to bottom are 1-3-4-2 in the period four region, 1-5-7-3-4-8-6-2 in the period 8 region as shown in the figure 4.

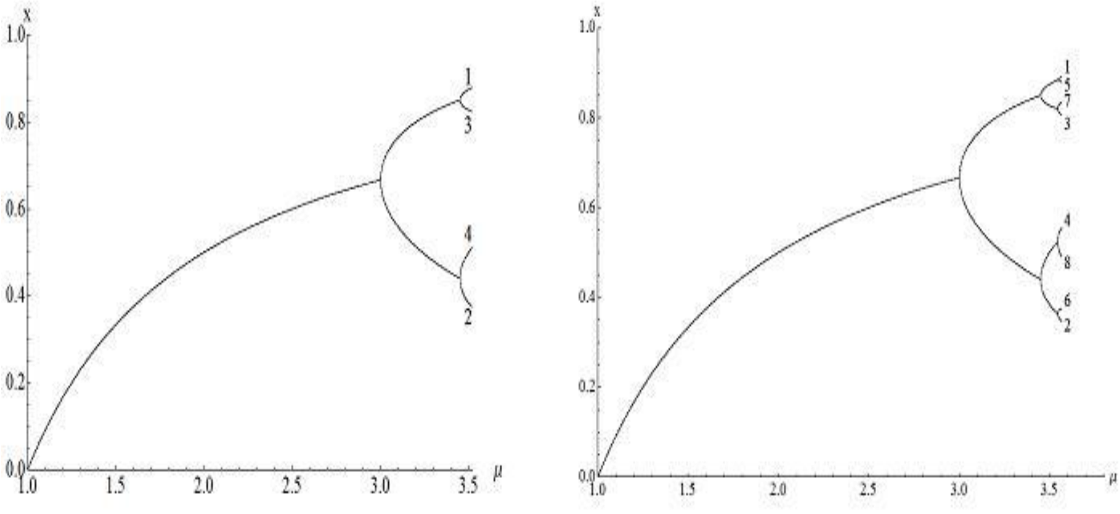

Figure 4

The Second Observation

It is not very difficult to recognize the heavy dark ‘curves’ of points that run through the chaotic region of the bifurcation diagram. These heavy concentrations of points are due to trajectories that pass near the critical point $x = 0.5$ of the iterated map function. All trajectories that pass near the critical point track each other for several subsequent iterations because the slope of the map function and all of its higher iterates is 0 at the critical point.

Thus, those trajectories diverge rather slowly leading to concentrations of points in the bifurcation diagram. The trajectory points that follow exactly from the critical point value x_c are called the images of the critical point. These images of the extrema, or critical points, of $f(x_n)$, are called the boundaries of the map. Analysis of boundaries helps to explain many diverse aspects of nonlinear maps, especially probability distributions in chaotic regions, the emergence of periodic orbits in such regions, and the effects of crises etc.

This description will enable us to better understand the reverse bifurcation or band splitting bifurcation phenomena, the Misiurewicz points [13] of the logistic map.

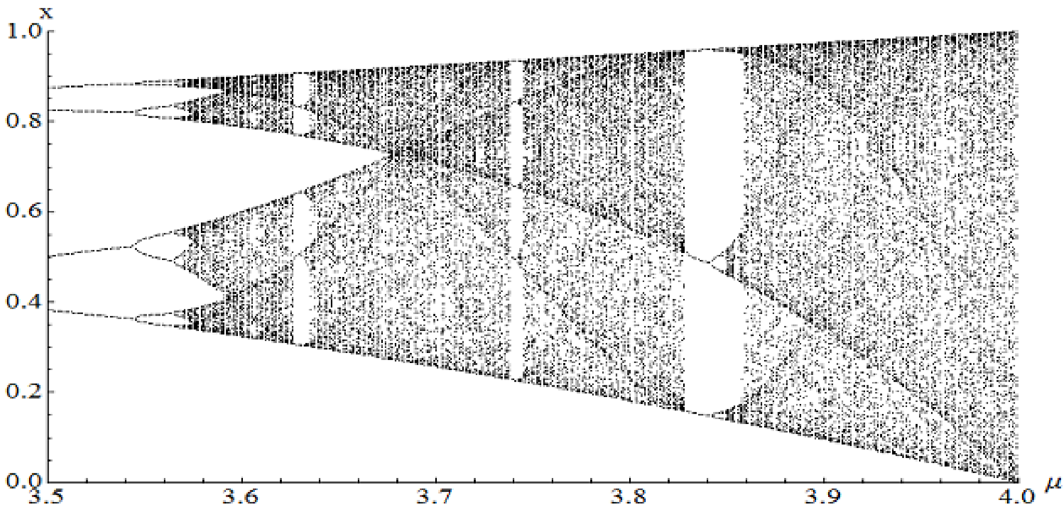

Figure 5: Bifurcation Diagram of the Logistic Map for $3.5 \leq \mu \leq 4$

Third Observation

The logistic map defined by the equation (1) is a inverted parabola with a maximum at $x = 0.5$. For a given μ, the function cannot attain a value greater than $\frac{\mu}{4}$. Therefore the long time dynamics of the map are confined to the interval $[f(\mu/4), \mu/4]$ which is $[f^2(x_c), f(x_c)]$.

$x_{max} = f(x_c)$ is the maximum x value visited by trajectories on the attractor for a particular parameter value.

$x_{min} = f(x_{max}) = f^{(2)}(x_c)$ is the minimum value of x visited for that parameter value.

This means, the first two iterates of f, starting from x_c, give the upper and lower limits for the attracting region.

For the logistic map, $x_c = \frac{\mu}{4}$, so the upper boundary of the chaotic bands is a straight line that hits $x = 1$ at $\mu = 4$. Also, $f^2(x_c)$ is given by $\left(\frac{\mu^2}{4}\right)\left(1 - \frac{\mu}{4}\right)$. Thus, the lower boundary of the chaotic bands is a cubic curve that hits $x = 0$ at $= 4$.

The dynamics of logistic map do not fill the entire unit interval until $\mu = 4$. The subsequent images of the critical point are interior boundaries which corresponds to the images of the critical point of the higher order iterates of the logistic map.

In figure 6, we have plotted the first 10 boundaries (orbit of the critical point [23]) (labelled in the figure) of the logistic map as a function of μ for $3.5 \leq \mu \leq 4.0$, and superimposed this graph on the bifurcation plot of the same region which was shown in figure 5 to create the figure 7.

For $\mu < \mu_\infty$, the accumulation point for the period doubling bifurcations, the interior boundaries confine the periodic orbits. For $\mu > \mu_\infty$, the boundaries not only confine the chaotic dynamics of the map, but also correspond exactly to the regions of high density.

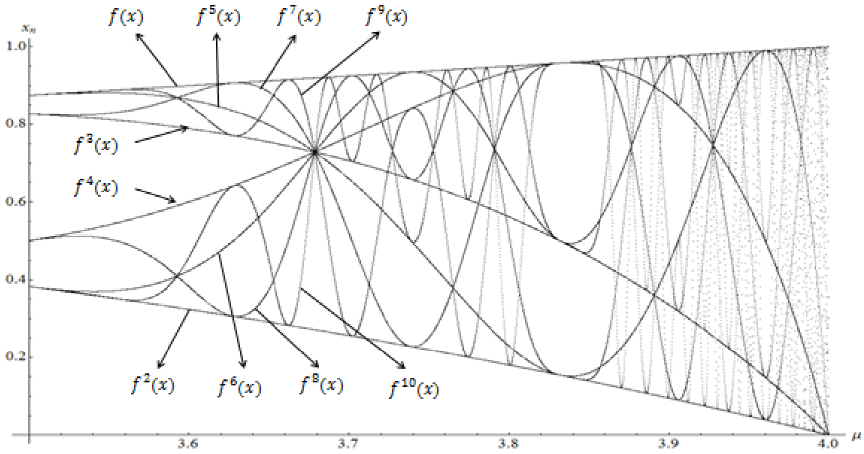

Figure 6: First Ten Images of the Critical Points in the Range $3.5 \leq \mu \leq 4$

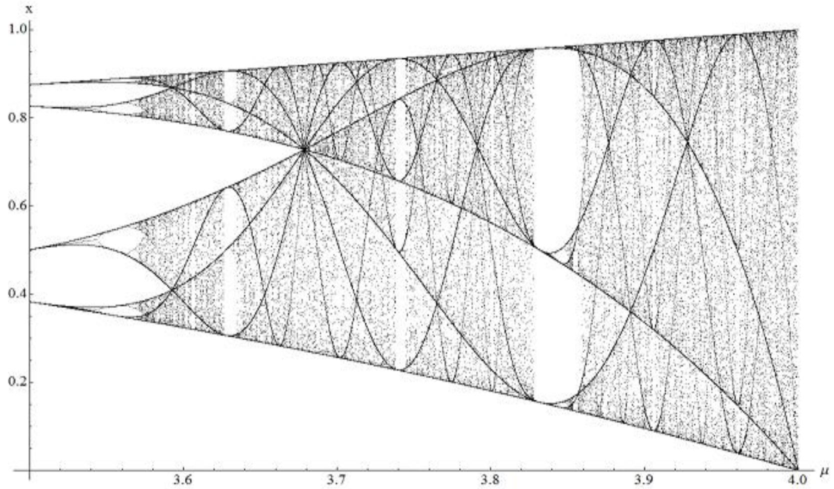

Figure 7: The Location of the First Ten Images of the Critical Point (Boundaries) are Plotted on the Bifurcation Diagram

It is important to note that the boundaries deviate from the bifurcation plot during periodic cycles, (shown in figure 5) since the dynamics of such cycles are governed by fixed points and not boundaries. But in regions where the map is chaotic, the boundaries form a skeletal frame which gives shape to the map. For $\mu > \mu_\infty$ these images of the critical point do delimit a set of chaotic "bands" to which the trajectorie are confined.

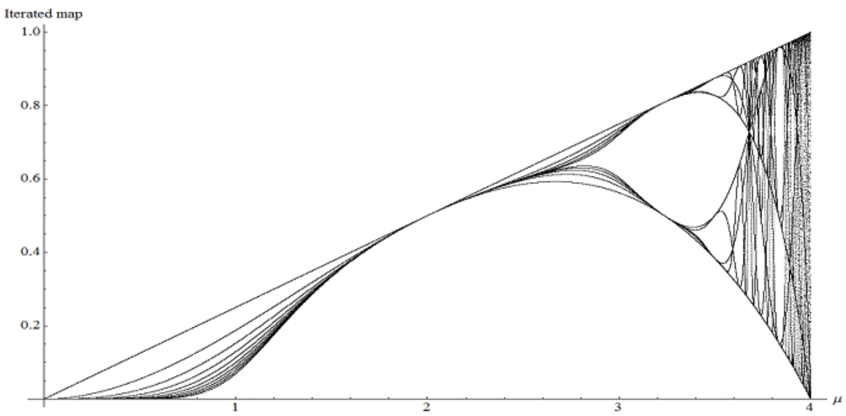

Figure 8: First Ten Images of the Critical Points in the Range $0 \leq \mu \leq 4$

REVERSE BIFURCATION OR BAND SPLITTING BIFURCATION AND MISIUREWICZ POINTS OR POINTS OF REVERSE BIFURCATION

From figure 5, 6 and 7, it is observed that near $\mu = 3.68$ two chaotic bands merge into one and there seems to be a convergence of the curves of the image points. This special image point is called the Misiurewicz point [13, 14, 15, 18].

We can notice that same type of crossing or convergence occurs where four chaotic bands merge to give two chaotic bands near $\mu = 3.6$.

If we observe the dynamics of the map from higher parameter values to lower ones, we can say that near $\mu = 3.68$ one chaotic band splits into two chaotic bands for lower parameter values. This phenomenon is called the reverse bifurction or the band splitting bifurcation [4, 8, 10].

We already observed that the first and the second iterates of the critical point constituted the outer boundaries of the iterative process after a possible transient period. In fact it is well known that the behaviour of the iterates of any function at its critical points is such that the first and second iterates are the upper and lower limits of the whole iterates or in otherwords any higher order iterate of the function will lie between these upper and lower limits.

Also in the region from μ_∞ to the Misiurewicz point where two chaotic bands merge into one, the inner most iterates will be the third iterate and the fourth iterate of the map function at its critical point, in which $f^3(x_c)$ is the lower boundary of the upper chaotic band and $f^4(x_c)$ is the upper boundary of the lower chaotic band [Fig. 6]. In between μ_∞ and the Misiurewicz point which is created during the over lapping of the two chaotic bands into one, there may be a large number of separate bands (say, approximately 2^∞bands near μ_∞) which merge together as the parameter varies and finally becomes a single band at the above mentioned Misiurewicz point. This band merging process takes place only when an unstable fixed point (say P_u) hits the attractor[4]. Generally speaking, the band merging of order 2^n to 2^{n-1} with n=1,2,3… takes place only when the unstable fixed points, which created during the bifurcation of 2^{n-1} cycle to 2^n cycle , hits the chaotic attractor or band of order 2^n [5]. Below we have shown this fact in case of the first three Misiurewicz points.

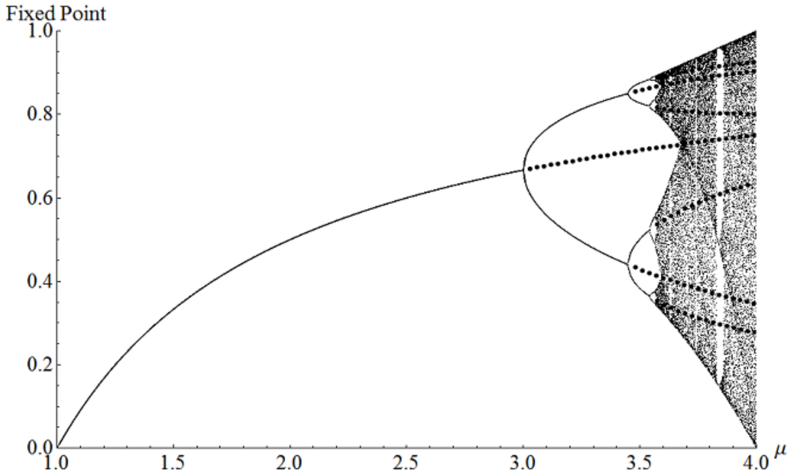

Figure 9: The Dotted Line Shows the Unstable Periodic Orbit Created at Every Period Doubling Bifurcation. It is to be Noted that Corresponding to Every Period Doubling Bifurcation there Exists a Misiurewicz Point

To the right of the Feigenbaum point or accumulation point, we find a completely different zone (formed by apparently chaotic bands) but at the same time with a lot of similarities to that of the left zone. In the right extreme, for $\mu = 4$, there is only one band spanning the whole interval from 0 to 1. It is the so called 1-periodic chaotic band. When μ decreases the band narrows. At $\mu = m_1$(say) the band splits into two parts that compose the $2(= 2^1)$ chaotic band. At $\mu = m_2$(say) the two bands split into four parts that compose the $4(= 2^2)$ chaotic bands and so on. Therefore, there is also a period doubling cascade of chaotic bands that finishes from the opposite side at the Feigenbaum point $\mu_\infty = m_\infty = 3.56994567 \ldots$[16]. This region is called the chaotic region, and points m are called band merging points or Misiurewicz points .

Technique to Find Out the Misiurewicz Points

In figure 6 and 7, we already observed that the first Misiurewicz point is the intersection of $f^3(x_c)$ *and* $f^4(x_c)$ through which all the other higher iterates of the map at its critical point passes. Thus, we can find the first Misiurewicz point by finding out the point of intersection of $f^3(x_c)$ *and* $f^4(x_c)$. Below we have furnished a schematic diagram (Figure 10) by which some of the next Misiurewicz points can be found out.

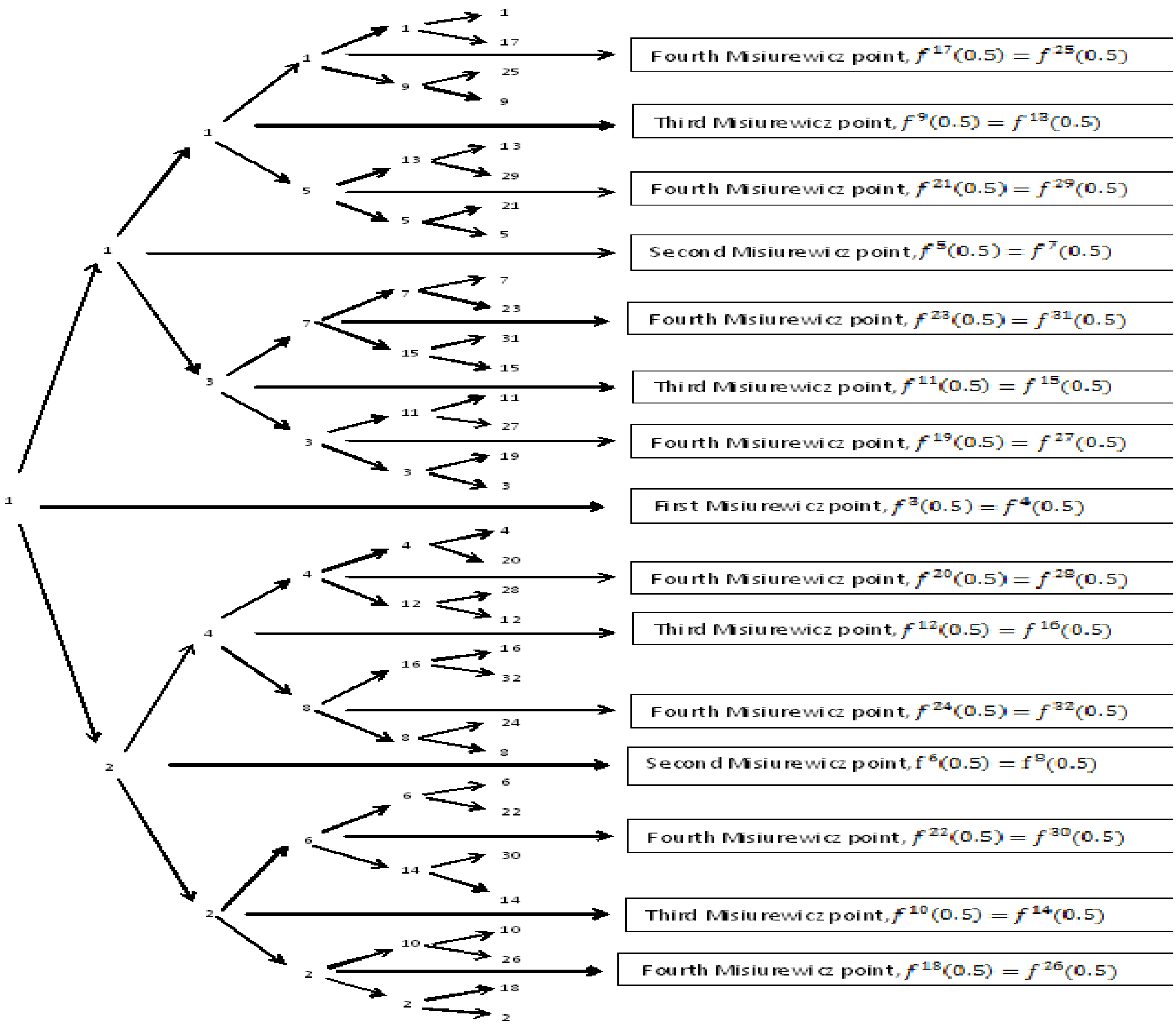

Figure 10

In fact, if we denote a Misiurewicz point as $M_{n,p}$ in the context that they have a preperiod n (they are n preperiodic) and they eventually have a period p (they are eventually preperiodic) then the i-th Misiurewicz point m_i can be represented by $M_{2^i+1,2^{i-1}}$, $i = 0, 1, 2, \ldots$ which can be found out from the point of intersection of $f_\mu^n\left(\frac{1}{2}\right)$ and $f_\mu^{n+p}\left(\frac{1}{2}\right)$

There is only one apparent exception : If we apply this rule for m_0 we shall obtain $M_{2,1/2}$, i.e two preperiodic and eventually $1/2$ periodic. But ½-periodic is 1-periodic, then it is $M_{2,1}$, as it really is [16].

In fact the band merging can be calculated analytically from the relation. $f^2(x_c, \mu_b) = \frac{1}{\mu}$ which we have shown below:

In the case of Logistic map, we have

$$x_c = \frac{1}{2},$$

$$f^2(x_c) = \mu f(x_c)(1 - f(x_c)),$$

$$f^3(x_c) = \mu f^2(x_c)(1 - f^2(x_c)) \text{ and}$$

$$f^4(x_c) = \mu f^2(x_c)(1 - f^2(x_c))(1 - \mu f^2(x_c)(1 - f^2(x_c))).$$

Hence the first band merging point is the point at which

$$f^4(x_c) - f^3(x_c) = 0$$

While solving this equation, we get

$$f^2(x_c) = \frac{\mu - 1}{\mu} \text{ and } \frac{1}{\mu}$$

Since $\frac{\mu-1}{\mu}$ is the unstable 1-cycle fixed point, the band merging crisis point is the point at which $f^2(x_c) = \frac{1}{\mu}$.

The equation $f^2(x_c) = \frac{1}{\mu}$ finally reduces to

$$\mu^3 - 2\mu^2 - 4\mu - 8 = 0$$

The only real solution to this equation is

$$\mu = \frac{2}{3} + \frac{1}{3}\left(152 - 24\sqrt{33}\right)^{\frac{1}{3}} + \frac{2}{3}\left(19 + 3\sqrt{33}\right)^{\frac{1}{3}}$$

$$= 3.678573510428\ldots$$

In the similar way using the formula $f^{2^i+1}\left(\frac{1}{2}\right) = f^{2^i+1+2^{i-1}}\left(\frac{1}{2}\right)$ and with the help of computer programming we have determined the reversed bifurcation point or Misiurewicz points as shown in the table below:

Table 2

Number of chaotic bands	Misiurewicz points (m_i)	Feigenbaum delta
First 2 chaotic band is born	3.678573510428322…	
First 4 chaotic band is born	3.592572184106978…	
First 8 chaotic band is born	3.574804938759208…	4.834437020275166…
First 16 chaotic band is born	3.570985940341614…	4.657119739532333…
First 32 chaotic band is born	3.570168472496375…	4.671741451276784…
First 64 chaotic band is born	3.569993388559133…	4.669005381415207…
First 128 chaotic band is born	3.569955891325219…	4.669249408391745…
First 256 chaotic band is born	3.569947860564655…	4.669200828940034…
First 512 chaotic band is born	3.569946140622108…	4.66920282584272…
First 1024 chaotic band is born	3.569945772263088…	4.669201767512598…

In the context of above calculations, it is to be noted that we get $p-1$ equations for every values of i which gives the same bifurcation value. So we can set the formula

$$f^{2^i+1}\left(\frac{1}{2}\right)=f^{2^i+1+2^{i-1}}\left(\frac{1}{2}\right)$$

As $f^{2^i+1+j}\left(\frac{1}{2}\right)=f^{2^i+1+2^{i-1}+j}\left(\frac{1}{2}\right)$ where $j=0,1,\dots p-1$ for every values of i.

For example for the first reverse bifurcation point, the equation is $f^3\left(\frac{1}{2}\right)=f^4\left(\frac{1}{2}\right)$

For second Misiurewicz point as $p=2$, so $j=0,1$ hence we get two equations viz

$$f^5\left(\frac{1}{2}\right)=f^7\left(\frac{1}{2}\right) \quad \text{and} \quad f^6\left(\frac{1}{2}\right)=f^8\left(\frac{1}{2}\right)$$

Both these equations gives the same value 3.574804938759208

Accumulation Point

Let $\{\mu_n\}$ be the sequence of bifurcation points. Using Feigenbaum δ, if μ_1,μ_2 are known then μ_3 can be predicted as $\mu_3=\frac{\mu_2-\mu_1}{\delta}+\mu_2$. Similarly $\mu_4=\frac{\mu_3-\mu_2}{\delta}+\mu_3$ which implies $\mu_4=(\mu_2-\mu_1)\left(\frac{1}{\delta}+\frac{1}{\delta^2}\right)+\mu_2$, repeating this argument it can be seen that $\mu_\infty=\frac{\mu_2-\mu_1}{\delta-1}+\mu_2$. However this expression is exact when $\delta_n=\frac{\mu_n-\mu_{n-1}}{\mu_{n+1}-\mu_n}$, the bifurcation ratio, is equal for all values of n. In fact $\{\delta_n\}$ converges as $n\to\infty$, i.e. $\lim_{n\to\infty}\delta_n=\delta$, So, we consider the sequence $\{\mu_{\infty,n}\}$, $\mu_{\infty,n}=\frac{\mu_n-\mu_{n-1}}{\delta-1}+\mu_n$, where μ_n are the experimental value of bifurcation points, clearly $\lim_{n\to\infty}\mu_{\infty,n}=\mu_\infty$,

Using the experimental bifurcation points the sequence of accumulation points $\{\mu_{\infty,n}\}$ is calculated for some values of n. The values are as follows:

$\mu_{\infty,1}=3.571993161965258\dots$

$\mu_{\infty,2}=3.569872703191509\dots$

$\mu_{\infty,3}=3.5699444125125472\dots$

$\mu_{\infty,4} = 3.5699455504551323\ldots$

$\mu_{\infty,5} = 3.5699456678654817\ldots$

$\mu_{\infty,6} = 3.5699456716448745\ldots$

$\mu_{\infty,7} = 3.569945671861678\ldots$

$\mu_{\infty,8} = 3.5699456718704905\ldots$

$\mu_{\infty,9} = 3.5699456718709253\ldots$

The above sequence converges to the value $3.5699456718709\ldots$, which is the required accumulation point .

For reverse bifurcation

$\mu_{\infty,1} = 3.569156752868087\ldots$

$\mu_{\infty,2} = 3.5699576900084806\ldots$

$\mu_{\infty,3} = 3.5699451148911456\ldots$

$\mu_{\infty,4} = 3.569945680747027\ldots$

$\mu_{\infty,5} = 3.5699456713835964\ldots$

$\mu_{\infty,6} = 3.569945671872079\ldots$

$\mu_{\infty,7} = 3.5699456718703715\ldots$

$\mu_{\infty,8} = 3.5699456718709417\ldots$

$\mu_{\infty,9} = 3.5699456718709577\ldots$

The above sequence converges to the value $3.5699456718709\ldots$, which is the same accumulation point (correct up to 13 decimal places).

CONCLUSIONS

From the above discussion, it becomes clear that the Feigenbaum point or the accumulation point can be considered as a chaotic mirror. The image of the point μ_0(where the one periodic orbit is born) is the point m_0 (where the 1-periodic chaotic band is born), the image of the point μ_1(where the 2-periodic orbit is born) is the point m_1(where the 2-periodic chaotic band is born), and so on. Periodic orbits in the Periodic region are images of chaotic bands in the chaotic region.

The Feigenbaum point or the accumulation point is at the same time m_∞ and μ_∞. In the first case, it is a very special Misiurewicz point with infinite number of chaotic bands and in the second case, it is a periodic point but with period ∞. It is then a very special separator.

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